数理解析研究所講究録
986 巻 1997 年 57-65



# Singular parts of moduli spaces for cubic polynomials and quadratic rational maps

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## 1. Quadratic rational maps

### 1.1. Moduli space of quadratic rational maps

Let $\overline{\mathbf{C}}$ be the Riemann sphere and $\mathrm{Rat}_2(\mathbf{C})$ the space of all quadratic rational maps from $\overline{\mathbf{C}}$ to itself. The group $\mathrm{PSL}_2(\mathbf{C})$ of Möbius transformations acts on the space $\mathrm{Rat}_2(\mathbf{C})$ by conjugation,

$$g \circ f \circ g^{-1} \in \mathrm{Rat}_2(\mathbf{C}) \quad \text{for} \quad g \in \mathrm{PSL}_2(\mathbf{C}),\ f \in \mathrm{Rat}_2(\mathbf{C}).$$

Two maps $f_1, f_2 \in \mathrm{Rat}_2(\mathbf{C})$ are **holomorphically conjugate**, denoted by $f_1 \sim f_2$, if and only if there exists $g \in \mathrm{PSL}_2(\mathbf{C})$ with $g \circ f_1 \circ g^{-1} = f_2$. The quotient space of $\mathrm{Rat}_2(\mathbf{C})$ under this action will be denoted by $\mathcal{M}_2(\mathbf{C})$, and called the **moduli space** of holomorphic conjugacy classes $\langle f \rangle$ of quadratic rational maps $f$.

Milnor introduced coordinates in $\mathcal{M}_2(\mathbf{C})$ as follows; for each $f \in \mathrm{Rat}_2(\mathbf{C})$, let $z_1, z_2, z_3$ be the fixed points of $f$ and $\mu_i$ the multipliers of $z_i$; $\mu_i = f'(z_i)$ $(1 \le i \le 3)$. Consider the elementary symmetric functions of the three multipliers,

$$\sigma_1 = \mu_1 + \mu_2 + \mu_3, \quad \sigma_2 = \mu_1\mu_2 + \mu_2\mu_3 + \mu_3\mu_1, \quad \sigma_3 = \mu_1\mu_2\mu_3.$$

These three multipliers determine $f$ up to holomorphic conjugacy, and are subject only to the restriction that

$$\sigma_3 = \sigma_1 - 2.$$

Hence the moduli space $\mathcal{M}_2(\mathbf{C})$ is canonically isomorphic to $\mathbf{C}^2$ with coordinates $\sigma_1$ and $\sigma_2$ (Lemma 3.1 in [Mil93]).

For each $\mu \in \mathbf{C}$ let $\mathrm{Per}_n(\mu)$ be the set of all conjugacy classes $\langle f \rangle$ of maps $f$ which having a periodic point of period $n$ and multiplier $\mu$.

Each of $\mathrm{Per}_1(\mu)$ and $\mathrm{Per}_2(\mu)$ forms a straight lines as follows:

$$\mathrm{Per}_1(\mu) = \left\{ \langle f \rangle \in \mathcal{M}_2(\mathbf{C}); \sigma_2 = (\mu + \mu^{-1})\sigma_1 - (\mu^2 + 2\mu^{-1}) \right\}$$
$$\mathrm{Per}_2(\mu) = \{ \langle f \rangle \in \mathcal{M}_2(\mathbf{C}); \sigma_2 = -2\sigma_1 + \mu \},$$

(Lemmas 3.4 and 3.6 in [Mil93]).

**Remark** $\mathrm{Per}_1(-1) \subseteq \mathrm{Per}_2(1)$ by definition. But, in the case of $\mathcal{M}_2(\mathbf{C})$, it is clear that two families coincide.

By an automorphism of a quadratic rational map $f$, we will mean $g \in \mathrm{PSL}_2(\mathbf{C})$ which commutes with $f$. The collection $\mathrm{Aut}(f)$ of all automorphisms of $f$ forms a finite group. It is clear that $\mathrm{Aut}(\tilde{f})$ is isomorphic to $\mathrm{Aut}(f)$ for any $\tilde{f} \in \langle f \rangle$.

The set

$$\mathcal{S} = \{\langle f \rangle\,;\ \mathrm{Aut}(f) \text{ is non-trivial}\} \subset \mathcal{M}_2(\mathbf{C})$$

is called the **symmetry locus**.

**Proposition 1** *The symmetry locus $\mathcal{S}$ of quadratic rational maps forms an irreducible algebraic curve as follows;*

$$S(\sigma_1, \sigma_2) = 2\sigma_1^3 + \sigma_1^2\sigma_2 - \sigma_1^2 - 4\sigma_2^2 - 8\sigma_1\sigma_2 + 12\sigma_1 + 12\sigma_2 - 36 = 0. \qquad (1)$$

**Proof of Corollary 1.**

$\mathrm{Aut}(f)$ coincides with the group consisting of all permutations of the fixed points which preserve the multipliers. In the case of $f$ has the three distinct fixed points, $\mathrm{Aut}(f)$ has order 1, 2, or 6 according as three multipliers are distinct, two are equal, or all the three are equal, respectively, while, if $f$ has multiple fixed points then $\mathrm{Aut}(f)$ is non-trivial if and only if $f$ has a triple fixed point. The multipliers $\mu_i$ are the roots of the equation:

$$\mu^3 - \sigma_1\mu^2 + \sigma_2\mu - \sigma_1 + 2 = 0. \qquad (2)$$

The equation (2) has multiple roots if and only if its discriminant is equal to zero. Hence we have

$$(\sigma_2 - 2\sigma_1 + 3)(2\sigma_1^3 + \sigma_1^2\sigma_2 - \sigma_1^2 - 4\sigma_2^2 - 8\sigma_1\sigma_2 + 12\sigma_1 + 12\sigma_2 - 36) = 0.$$

The first factor corresponds with $\mathrm{Per}_1(1)$. Considering the line of the first factor ($\mathrm{Per}_1(1)$) tangent to the curve of the second factor ($S$) with tangency of degree three, the second factor is the required equation.

∎

The following result is obtained immediately by the definition of the envelope of the family of curves.

**Corollary 1** *The envelope of $\{Per_1(\mu)\}_\mu$ coincides with the symmetry locus.*

**Remark** (Theorem 5.1.of [Mil93]) A quadratic rational map has a non-trivial automorphism if and only if it is conjugate to a map in the unique normal form $f(z) = k(z + \frac{1}{z})$ with $k \in \mathbf{C} \setminus \{0\}$.

### 1.2. Real moduli space

Let $\mathrm{Rat}_2(\mathbf{R})$ be the set of real quadratic rational maps. Then the parameters $\sigma_i$ ($1 \le i \le 3$) are all real, because the three fixed points and the corresponding multipliers are either all real or one real and a pair of complex conjugate numbers. According to J. Milnor, we define the real moduli space $\mathcal{M}_2(\mathbf{R})$ for $\mathrm{Rat}_2(\mathbf{R})$ to be simply the real $(\sigma_1, \sigma_2)$-plane. This notation needs some care when used: if we put $\mathcal{S}_{\mathbf{R}} = \mathcal{S} \cap \mathcal{M}_2(\mathbf{R})$, and denote by $\langle\ \rangle_{\mathbf{R}}$ the real conjugacy class, then $(\mathrm{Rat}_2(\mathbf{R})/\mathrm{PGL}_2(\mathbf{R})) \setminus \{\langle a(x+\frac{1}{x})\rangle_{\mathbf{R}}, \langle a(x-\frac{1}{x})\rangle_{\mathbf{R}}\}_{a\in\mathbf{R}^\times}$ is canonically isomorphic to $\mathbf{R}^2 \setminus \mathcal{S}_{\mathbf{R}}$, whereas there is a canonical two-to-one correspondence between $\{\langle a(x \pm \frac{1}{x})\rangle\}_{a\in\mathbf{R}^\times}$ and $\mathcal{S}_{\mathbf{R}}$.

For map $f \in \mathcal{M}_2(\mathbf{R})$, the two critical points of $f$ are two real numbers or a pair of complex conjugate numbers. If $f$ has a pair of complex conjugate critical points, this map is two-to-one covering map on $S^1 = \mathbf{R} \cup \{\infty\}$. In this case, if $f' > 0$ then $f$ is called the map of degree $+2$, else $f' < 0$ then the map of degree $-2$.

While a map $f$ with real critical points is called monotone (resp. unimodal, bimodal) if the interval $I = int(f(S))$ contains no (resp. one, two) critical points ([Mil93]).

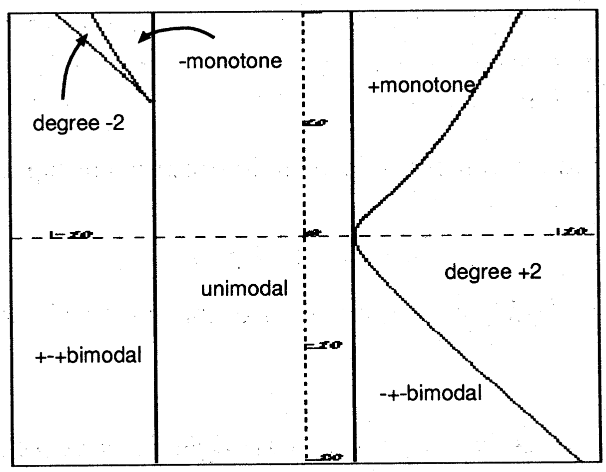


Fig. 1. The topological partition of the $\mathcal{M}_2(\mathbf{R})$.

**Boundary curves of Figure 1**

$$\begin{aligned} CD_1 &: \sigma_1 = 2 \\ BC_1 &: \sigma_1 = 6 \\ \text{Symmetry locus} &: S(\sigma_1, \sigma_2) = 0 \end{aligned}$$

where the curves $CD_1$ ($\mathrm{Per}_1(0)$) and $BC_1$ are "center curve" defined in [NN93].

**Remark** Two curves $BC_1$ and $CD_1$ are boundary curves of the "unimodal" region.

## 2. Cubic polynomials

### 2.1. Moduli space of cubic polynomials

Let $\mathrm{Poly}_3(\mathbf{C})$ be the space of all cubic polynomials from $\mathbf{C}$ to itself. The group $\mathrm{Poly}_1(\mathbf{C})$ of affine transformations acts on the space $\mathrm{Poly}_3(\mathbf{C})$ by conjugation,

$$g \circ p \circ g^{-1} \in \mathrm{Poly}_3(\mathbf{C}) \quad \text{for} \quad g \in \mathrm{Poly}_1(\mathbf{C}),\ p \in \mathrm{Poly}_3(\mathbf{C}).$$

Two maps $p_1, p_2 \in \mathrm{Poly}_3(\mathbf{C})$ are **holomorphically conjugate**, denoted by $p_1 \sim p_2$, if and only if there exists $g \in \mathrm{Poly}_1(\mathbf{C})$ with $g \circ p_1 \circ g^{-1} = p_2$. The quotient space of $\mathrm{Poly}_3(\mathbf{C})$ under this action will be denoted by $\mathrm{M}_3(\mathbf{C})$, and called the **moduli space** of holomorphic conjugacy classes $\langle p \rangle$ of cubic polynomials $p$.

Doing the same as the case of quadratic rational maps, we introduce coordinates in $\mathrm{M}_3(\mathbf{C})$ as follows; for each $p \in \mathrm{Poly}_3(\mathbf{C})$, let $z_1,\ z_2,\ z_3,\ z_4(=\infty)$ be the fixed points of $p$ and $\mu_i$ the multipliers of $z_i$; $\mu_i = p'(z_i)$ $(1 \le i \le 3)$, and $\mu_4 = 0$. Consider the elementary symmetric functions of the four multipliers,

$$\begin{aligned}
\sigma_1 &= \mu_1 + \mu_2 + \mu_3 + \mu_4 = \mu_1 + \mu_2 + \mu_3 \\
\sigma_2 &= \mu_1\mu_2 + \mu_1\mu_3 + \mu_1\mu_4 + \mu_2\mu_3 + \mu_2\mu_4 + \mu_3\mu_4 = \mu_1\mu_2 + \mu_1\mu_3 + \mu_2\mu_3 \\
\sigma_3 &= \mu_1\mu_2\mu_3 + \mu_1\mu_2\mu_4 + \mu_1\mu_3\mu_4 + \mu_2\mu_3\mu_4 = \mu_1\mu_2\mu_3 \\
\sigma_4 &= \mu_1\mu_2\mu_3\mu_4 = 0.
\end{aligned}$$

These multipliers determine uniquely $p$ up to holomorphic conjugacy, and are subject only to the restriction that

$$3 - 2\sigma_1 + \sigma_2 = 0.$$

Hence the moduli space $\mathrm{M}_3(\mathbf{C})$ is canonically isomorphic to $\mathbf{C}^2$ with coordinates $\sigma_1$ and $\sigma_3$.

**Proposition 2** *The locus $Per_1(\mu)$ forms a straight lines as follows:*

$$Per_1(\mu) = \left\{ \langle f \rangle \in M_3(\mathbf{C}); \sigma_3 = (-\mu^2 + 2\mu)\sigma_1 + \mu^3 - 3\mu \right\}.$$

*The locus $Per_2(\mu)$ forms an algebraic curve of degree three as follows:*

$$\begin{aligned}
Per_2(\mu) \;=\; & \{\langle f \rangle \in \mathcal{M}_2(\mathbf{C}); \sigma_3^2 + (4\sigma_1^2 - (\mu + 57)\sigma_1 + 252)\sigma_3 - (4\mu - 16)\sigma_1^3 \\
& +(61\mu - 252)\sigma_1^2 - (4\mu^2 + 246\mu - 1134)\sigma_1 - \mu^3 + 51\mu^2 \\
& -99\mu - 459 = 0\}.
\end{aligned}$$

*Note that this curve is irreducible if and only if $\mu \neq 1$. In the case of $\mu = 1$,*

$$Per_2(1) = Per_1(-1) \bigcup \left\{ \langle f \rangle \in \mathcal{M}_2(\mathbf{C}); \sigma_3 + 4\sigma_1^2 - 61\sigma_1 + 254 = 0 \right\}.$$

Using conjugation described in above, we can define symmetry locus of this moduli space as one in $\mathcal{M}_2(\mathbf{C})$, and we obtain next results.

**Theorem 1** *The symmetry locus $\mathcal{S}$ of cubic polynomials forms an irreducible algebraic curve:*

$$S(\sigma_1, \sigma_3) = 27\sigma_3 + (\sigma_1 - 6)(2\sigma_1 - 3)^2 = 0. \tag{3}$$

The following result is obtained immediately by the definition of the envelope of the family of curves.

**Corollary 2** *The envelope of $\{Per_1(\mu)\}_\mu$ coincides with the symmetry locus.*

**Remark** A cubic polynomial has non-trivial automorphism if and only if it is conjugate to a map in the unique normal form $p(z) = z^3 + az$.

### 2.2. Real moduli space

Let $\mathrm{Poly}_3(\mathbf{R})$ be the set of real cubic polynomials. By the same reason for the case of $\mathcal{M}_2$, we define the real moduli space $\mathrm{M}_3(\mathbf{R})$ for $\mathrm{Poly}_3(\mathbf{R})$ to be simply the real $(\sigma_1, \sigma_3)$-plane. This notation needs some care when used: if we put $\mathcal{S}_\mathbf{R} = \mathcal{S} \cap \mathrm{M}_3(\mathbf{R})$, and denote by $\langle\ \rangle_\mathbf{R}$ the real conjugacy class, then $(\mathrm{Poly}_3(\mathbf{R})/\mathrm{Poly}_1(\mathbf{R})) \setminus \{\langle x^3 + ax\rangle_\mathbf{R}, \langle -x^3 + ax\rangle_\mathbf{R}\}_{a \in \mathbf{R}^\times}$ is canonically isomorphic to $\mathbf{R}^2 \setminus \mathcal{S}_\mathbf{R}$, whereas there is a canonical two-to-one correspondence between $\{\langle \pm x^3 + ax\rangle\}_{a \in \mathbf{R}^\times}$ and $\mathcal{S}_\mathbf{R}$.

For map $p \in \mathrm{M}_3(\mathbf{R})$, if the real filled-in Julia set of $p$ is a single point then it is said that $p$ in the class $\mathcal{R}_0$. Let $J$ be the smallest closed interval which contains the real filled-in Julia set of $p$. For $p \notin \mathcal{R}_0$, it is said that $p$ belongs to the class $\mathcal{R}_n$ if the graph of $p$ intersected with $J \times J$ has $n$ distinct components ([Mil92]).

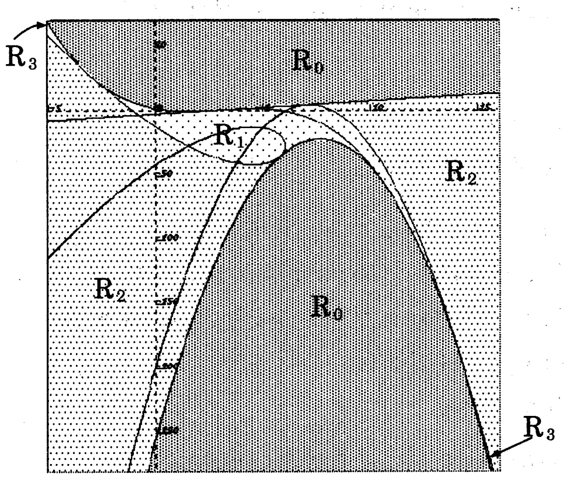


Fig. 2. The topological partition of the $\mathrm{M}_3(\mathbf{R})$.

**Boundary curves of Figure 2**

$$\begin{array}{rcl}
\mathrm{Per}_1(1) & : & \sigma_1 - \sigma_3 - 2 = 0 \\
\mathrm{Preper}_{(1)}1 & : & -4\sigma_1^2 + 57\sigma_1 - \sigma_3 - 198 = 0 \\
\text{Symmetry locus} & : & S(\sigma_1, \sigma_3) = 0 \\
\mathrm{Per}_2(1) & : & -8\sigma_1^3 + 180\sigma_1^2 - 1809\sigma_1 - 27\sigma_3 + 6966 = 0 \\
\mathrm{Preper}_{(1)}2 & : & 64\sigma_1^6 - 1152\sigma_1^5 + 7776\sigma_1^4 + (432\sigma_3 - 25056)\sigma_1^3 \\
& & +(-3888\sigma_3 + 41796)\sigma_1^2 + (8748\sigma_3 - 34992)\sigma_1 + 729\sigma_3^2 \\
& & -45198\sigma_3 + 543105 = 0
\end{array}$$

## 3. Polynomials of degree $n$

### 3.1. Moduli space of polynomials of degree $n$

Now we discuss about the moduli space $\mathrm{M}_n(\mathbf{C})$ for the space, $\mathrm{Poly}_n(\mathbf{C})$, of polynomials of degree $n$.

Doing the same as the case of cubic polynomials, we try introducing coordinates in $\mathrm{M}_n(\mathbf{C})$ as follows; for each $p(z) \in \mathrm{Poly}_n(\mathbf{C})$, let $z_1, \cdots, z_n, z_{n+1}(=\infty)$ be the fixed points of $p$ and $\mu_i$ the multipliers of $z_i$; $\mu_i = p'(z_i)$ $(1 \le i \le n)$, and $\mu_{n+1} = 0$. Consider the elementary symmetric functions of the $n$ multipliers,

$$\begin{aligned}
&\sigma_{n,1} = \mu_1 + \cdots + \mu_n, \\
&\sigma_{n,2} = \mu_1\mu_2 + \cdots + \mu_{n-1}\mu_n = \textstyle\sum_{i=1}^{n-1} \mu_i \sum_{j>i}^{n} \mu_j, \\
&\quad \cdots \\
&\sigma_{n,n} = \mu_1\mu_2 \cdots \mu_n, \\
&\sigma_{n,n+1} = 0.
\end{aligned}$$

**Example 1** For example, we assume $p(z) \in \mathrm{Poly}_4(\mathbf{C})$;

- fixed points: $z_1, z_2, z_3, z_4, \infty$
- multiplier: $\mu_1, \mu_2, \mu_3, \mu_4, 0$
- elementary symmetric functions:

$$\begin{cases}
\sigma_{4,1} = \mu_1 + \mu_2 + \mu_3 + \mu_4 \\
\sigma_{4,2} = \mu_1\mu_2 + \mu_1\mu_3 + \mu_1\mu_4 + \mu_2\mu_3 + \mu_2\mu_4 + \mu_3\mu_4 \\
\sigma_{4,3} = \mu_1\mu_2\mu_3 + \mu_1\mu_2\mu_4 + \mu_1\mu_3\mu_4 + \mu_2\mu_3\mu_4 \\
\sigma_{4,4} = \mu_1\mu_2\mu_3\mu_4 \\
\sigma_{4,5} = 0
\end{cases}$$

Applying Fatou-index theorem to these fixed points;

$$\frac{1}{1-\mu_1} + \frac{1}{1-\mu_2} + \frac{1}{1-\mu_3} + \frac{1}{1-\mu_4} + \frac{1}{1-0} = 1, \tag{4}$$

where $\mu_i \ne 1$ $(1 < i < n)$. Arranging this equation for the form of elementary symmetric functions;

$$\begin{aligned}
4 - 3(\mu_1 + \mu_2 + \mu_3 + \mu_4) + 2(\mu_1\mu_2 + \mu_1\mu_3 + \mu_1\mu_4 + \mu_2\mu_3 + \mu_2\mu_4 + \mu_3\mu_4) \\
-(\mu_1\mu_2\mu_3 + \mu_1\mu_2\mu_4 + \mu_1\mu_3\mu_4 + \mu_2\mu_3\mu_4) = 0.
\end{aligned}$$

Hence we have

$$4 - 3\sigma_{4,1} + 2\sigma_{4,2} - \sigma_{4,3} = 0. \tag{5}$$

For the equation (5), the cases $\mu_i = 1$ are also allowable.

Now we consider a polynomial $p(z) = a_4 z^4 + a_3 z^3 + a_2 z^2 + a_1 z + a_0 \in \mathrm{Poly}_4(\mathbf{C})$ that has at least two fixed points. After affine conjugation, we can assume they are 0 and 1. Then, we will solve the following question: *"Do the four multipliers*

$$\mu_0 = p'(0),\ \mu_1 = p'(1),\ \mu_2 = p'(z_2),\ \mu_3 = p'(z_3),$$

*where $z_1$, $z_2$ are fixed points of $p(z)$, determine the five coefficients $a_4, a_3, a_2, a_1, a_0$ of $p(z)$?"*

In fact, the following equations hold;

$$\begin{array}{ll} a_0 = 0 & \text{because of } f(0) = 0, \\ a_1 = \mu_0 & \text{because of } f'(0) = \mu_0, \\ a_2 = a_4 + 3 - 2\mu_0 - \mu_1 & \text{because of } f'(1) = \mu_1, \\ a_3 = 1 - a_4 - a_2 - \mu_0 & \text{because of } f(1) = 1, \end{array}$$

and $a_4$ is a common root of the following two equations;

$$\begin{aligned} A_1 =& (\mu_2^2 - 2\mu_3\mu_2 + \mu_3^2 - \mu_0^2 + 2\mu_1\mu_0 - \mu_1^2)a_4^4 + (-4\mu_0^3 + (4\mu_1 + 8)\mu_0^2 + (-4\mu_1^2 - 8)\mu_0 + 4\mu_1^3 - 8\mu_1^2 + \\ & 8\mu_1)a_4^3 + (-6\mu_0^4 + (-4\mu_1 + 28)\mu_0^3 + (4\mu_1^2 + 4\mu_1 - 44)\mu_0^2 + (-4\mu_1^3 + 4\mu_1^2 - 8\mu_1 + 32)\mu_0 - 6\mu_1^4 + \\ & 28\mu_1^3 - 44\mu_1^2 + 32\mu_1 - 16)a_4^2 + (-4\mu_0^5 + (-12\mu_1 + 32)\mu_0^4 + (-8\mu_1^2 + 64\mu_1 - 96)\mu_0^3 + (8\mu_1^3 - \\ & 96\mu_1 + 128)\mu_0^2 + (12\mu_1^4 - 64\mu_1^3 + 96\mu_1^2 - 64)\mu_0 + 4\mu_1^5 - 32\mu_1^4 + 96\mu_1^3 - 128\mu_1^2 + 64\mu_1)a_4 - \mu_0^6 + \\ & (-6\mu_1 + 12)\mu_0^5 + (-15\mu_1^2 + 60\mu_1 - 60)\mu_0^4 + (-20\mu_1^3 + 120\mu_1^2 - 240\mu_1 + 160)\mu_0^3 + (-15\mu_1^4 + \\ & 120\mu_1^3 - 360\mu_1^2 + 480\mu_1 - 240)\mu_0^2 + (-6\mu_1^5 + 60\mu_1^4 - 240\mu_1^3 + 480\mu_1^2 - 480\mu_1 + 192)\mu_0 - \mu_1^6 + \\ & 12\mu_1^5 - 60\mu_1^4 + 160\mu_1^3 - 240\mu_1^2 + 192\mu_1 - 64 = 0, \\ A_2 =& (\mu_2 + \mu_3 + \mu_0 + \mu_1 - 4)a_4^2 + (2\mu_0^2 - 4\mu_0 - 2\mu_1^2 + 4\mu_1)a_4 + \mu_0^3 + (3\mu_1 - 6)\mu_0^2 + (3\mu_1^2 - 12\mu_1 + \\ & 12)\mu_0 + \mu_1^3 - 6\mu_1^2 + 12\mu_1 - 8 = 0. \end{aligned}$$

Above two equations have common roots if and only if $\mu_0, \mu_1, \mu_2, \mu_3$ satisfy the equation (5). Since $\mu_0, \mu_1, \mu_2, \mu_3$ are the four multipliers of $p(z)$ and they should satisfy the equation (5), the two equations always have common roots. Hence five coefficients of $p(z)$ are calculated by its four multipliers, however, this calculation is not decisive when they have distinct two common roots.

For the case of $\mathrm{Poly}_n(\mathbf{C})$, it is clear from (4) that the equation corresponds to (5) cannot have the term of $\sigma_{n,n}$. Hence we can put

$$c_0 + c_1\sigma_{n,1} + c_2\sigma_{n,2} + \cdots + c_{n-1}\sigma_{n,n-1} = 0$$

where $c_k$ $(0 \le k \le n-1)$ are functions of $n$ variable.

Paying attention to the form of elementary symmetric functions, we obtain the following equation;

$$c_k = (-1)^k \binom{n-1}{k} n \Big/ \binom{n}{k} = n - k.$$

where $\begin{pmatrix} n \\ k \end{pmatrix}$ means binomial coefficient. For convenience, put $\sigma_{n,0} = 1$. we have

$$\sum_{k=0}^{n-1}(-1)^k(n-k)\sigma_{n,k} = 0. \tag{6}$$

**Question** Is the moduli space $M_n(\mathbf{C})$ for polynomials of degree $n$ canonically isomorphic to $\mathbf{C}^{n-1}$ with coordinates $\sigma_1$, $\sigma_2$, $\cdots$, $\sigma_{n-2}$, and $\sigma_n$?

### 3.2. Symmetry locus

**Proposition 3** *A polynomial of degree four has a non-trivial automorphism if and only if it is conjugate to a map in the unique normal form*

$$\{z^4 + az\}, \quad a \in \mathbf{C}.$$

*For a map $p(z)$ in this normal form, $Aut(p)$ is a cyclic group of order three.*

**Outline of proof.** Let $p(z) \in \mathrm{Poly}_4(\mathbf{C})$.

1. In the case of a map $p(z)$ with multiple fixed points.
   (a) The case of $p(z)$ with a fixed point of order four: $\mathrm{Aut}(p)$ is non-trivial.
   (b) The case of $p(z)$ with a fixed point of order three: $\mathrm{Aut}(p)$ is trivial.
   (c) The case of $p(z)$ with two fixed points of order two: there is not such $p(z)$.
   (d) The case of $p(z)$ with a fixed point of order two: $\mathrm{Aut}(p)$ is trivial.
2. In the case of a map $p(z)$ with four distinct fixed points.
   (a) The case of four distinct multipliers: $\mathrm{Aut}(p)$ is trivial.
   (b) The case that only two of multipliers are coincide: $\mathrm{Aut}(p)$ is trivial.
   (c) The case of two pair of same multipliers: there is not such $p(z)$.
   (d) The case of three same multipliers: By an affine conjugation, if three fixed points (whose multipliers are same) are mapped on the vertices of a regular triangle whose barycenter is the origin and the other fixed point on the origin, then $\mathrm{Aut}(p)$ is non-trivial. Otherwise $\mathrm{Aut}(p)$ is trivial.
   (e) The case of four same multipliers: there is not such $p(z)$.

Therefore a map $p(z)$ has non-trivial automorphisms if and only if $p(z)$ is in the case 1-(a) and the first part of 2-(d). We can check easily that these maps coincide with the normal form $\{z^4 + az\}$. ∎

**Conjecture** A polynomial of degree $n$ has a non-trivial automorphism if and only if it is conjugate to a map in the unique normal form

$$\left\{ z^n + \sum_{k|(n-1), k \neq n-1} A(k) z^k \right\}$$

where $A(k)$ are parameters in $\mathbf{C}$.

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